



# Hyperboloid, instanton, oscillator

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## Abstract

We suggest the exactly solvable model of the oscillator on a four-dimensional hyperboloid which interacts with a  $SU(2)$  instanton. We calculate its wavefunctions and spectrum.

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## 1. Introduction

There exists a well-known generalization of the  $N$ -dimensional isotropic oscillator to spheres and hyperboloids, suggested by Higgs [1]. The uniqueness of this system is that it inherits all constants of motion of the standard (flat) oscillator, although its symmetry algebra becomes nonlinear (whereas the symmetry algebra of the flat oscillator is  $su(N)$ ). However, this system does not respect the inclusion of external gauge fields. For instance, the inclusion of a constant magnetic field in the two-dimensional Higgs oscillator breaks even its exact solvability. On the other hand, a constant magnetic field preserves the kinematical symmetries of a free particle on the two-dimensional sphere and hyperboloid (which form, respectively,  $so(3)$  and  $so(1,2)$  Lie algebras), as well as the exact solvability of the planar (two-dimensional) oscillator. Similarly, the inclusion of the BPST instanton field preserves the kinematical symmetries of a free particle moving on the four-dimensional sphere and the exact solvability of the four-dimensional flat oscillator, but it breaks the exact solvability of the Higgs oscillator on the four-dimensional sphere. Instead, the analog of the oscillator on both complex projective spaces  $\mathbb{C}P^N$  and their noncompact analogs, i.e., Lobachevsky spaces  $\mathcal{L}_N =$

$SU(N,1)/U(N)$  [2], loosing part of the hidden symmetries, remains, nevertheless, exactly solvable in the presence of a constant magnetic field. Since  $\mathbb{C}P^1$  is the two-dimensional sphere, and  $\mathcal{L}_1$  is the two-dimensional hyperboloid, the above model seems to be an appropriate alternative to the two-dimensional Higgs oscillator, as the analog of the planar oscillator with constant magnetic field. These systems are defined by the potential [2]

$$V_{\mathbb{C}P^N} = \omega^2 r_0^2 \frac{u_a \bar{u}_a}{u_0 \bar{u}_0},$$

$$u_0 \bar{u}_0 + \epsilon u_a \bar{u}_a = r_0^2, \quad a = 1, \dots, N, \quad \epsilon = \pm 1, \quad (1)$$

where  $u_a/u_0$  are homogeneous complex coordinates for  $\mathbb{C}P^N$  and  $\mathcal{L}_N$ ,  $\epsilon = 1$  corresponds to the  $\mathbb{C}P^N$ , and  $\epsilon = -1$  corresponds to the  $\mathcal{L}_N$ .

In [3] it was claimed, that the potential (1), would be the appropriate generalization of the oscillator on the quaternionic projective spaces with  $\mathbb{H}P^N$  respecting the inclusion of the (constant uniform) instanton field, provided we interpret  $\mathbf{u}_a$  and  $\mathbf{u}_0$  as quaternionic coordinates of the ambient quaternionic space  $\mathbb{H}^{N+1}$ . For  $\mathbb{H}P^1$  (i.e., for the four-dimensional sphere) it was shown that this is indeed the case [4]. Moreover, in contrast with the case of  $\mathbb{C}P^1$ , the spectrum of the system depends on the topological charge of the instanton (what might be connected to the behaviour of two-dimensional noncommutative quantum mechanics models in a constant magnetic field [5]). The invention and study of this model was motivated by the

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recently suggested theory of the four-dimensional Hall effect [6], which attracted a wide interest in the physics community (see, e.g., [7,8]). This theory is based on the quantum mechanics of colored particle moving on a four-dimensional sphere in the field of a  $SU(2)$  Yang monopole [10] (which is equivalent to the BPST instanton [11]). The inclusion of the potential reduced it to the effective three-dimensional edge theory. The key role in this model is played by the external instanton field, which provides it with a degenerate ground state, becoming infinite in the planar limit. This theory displays a few interesting phenomena, such as an infinite gapless sequence of massless particle excitations with any spin. The initial (without potential term) symmetry of this theory is  $SO(5)$ , and it has no relativistic interpretation. On the other hand, the quantum mechanics of the colored particle on the four-dimensional hyperboloid, moving in the (constant uniform) field of  $SU(2)$  instanton would possess a  $SO(4,1)$  symmetry. There is no doubt that it will have a degenerate ground state, hence, developing the theory of four-dimensional quantum Hall effect will be possible on this quantum-mechanical background too. One can expect that this hypothetic,  $SO(4,1)$ -symmetric theory of the four-dimensional Hall effect will have a relativistic interpretation. By this reason, the construction of the noncompact analog of the model [4], i.e., the exactly-solvable model of the oscillator on the four-dimensional hyperboloid interacting with the  $SU(2)$  instanton field, seems to be even more important, than the initial “compact” system. This is the subject of the present Letter.

The Letter is arranged as follows. In Section 2 we construct the hyperbolic analog of the BPST instanton and suggest the appropriate oscillator potential. In Section 3 we get the energy spectrum and wavefunctions of the system.

## 2. Instanton

In this section we construct the  $SU(2)$  instanton and anti-instanton on the four-dimensional two-sheet hyperboloid, which defines the field configuration with constant magnitude, i.e., the hyperbolic analog of the BPST instanton. Also, we present the hyperbolic analog of the oscillator potential on the four-dimensional sphere considered in Ref. [4]. It is convenient to get these basic ingredients of the model by the use of quaternions, following [9].

Let us parameterize the (ambient) pseudo-Euclidean space  $\mathbb{R}^{4,1}$  by the real coordinate  $x_0$  and the quaternionic one  $\mathbf{x} = x_4 + \sum_{a=1}^3 x_a \mathbf{e}_a$ , with  $\mathbf{e}_a \mathbf{e}_b = -\delta_{ab} + \varepsilon_{abc} \mathbf{e}_c$ ,  $\bar{\mathbf{e}}_a = -\mathbf{e}_a$ . Notice that  $t_a \equiv \mathbf{e}_a/2$  form a  $su(2)$  algebra:  $[t_a, t_b] = \varepsilon_{abc} t_c$ . In terms of these coordinates the metric on  $\mathbb{R}^{4,1}$  reads

$$ds^2 = d\mathbf{x} d\bar{\mathbf{x}} - dx_0^2. \quad (2)$$

Imposing the constraint

$$x_0^2 - \mathbf{x}\bar{\mathbf{x}} = r_0^2, \quad (3)$$

we shall get the metric on the four-dimensional hyperboloid. It is convenient to resolve this constraint, choosing

$$x_0 = r_0 \frac{1 + \mathbf{w}\bar{\mathbf{w}}}{1 - \mathbf{w}\bar{\mathbf{w}}}, \quad \mathbf{x} = r_0 \frac{2\mathbf{w}}{1 - \mathbf{w}\bar{\mathbf{w}}}, \quad (4)$$

where  $|\mathbf{w}| < 1$  for the upper sheet of hyperboloid, and  $|\mathbf{w}| > 1$  for the lower one. In these coordinates the metric (2), restricted to the hyperboloid, looks as follows:

$$ds^2 = \frac{4r_0^2 d\mathbf{w} d\bar{\mathbf{w}}}{(1 - \mathbf{w}\bar{\mathbf{w}})^2}. \quad (5)$$

This is precisely the quaternionic analog of the Poincaré model of the Lobachevsky plane (two-dimensional two-sheet hyperboloid). The instanton and anti-instanton solutions are defined by the following expressions:

$$\begin{aligned} \mathbf{A}_+ &= \text{Im} \frac{\mathbf{w} d\bar{\mathbf{w}}}{\mathbf{w}\bar{\mathbf{w}} - 1} = -\text{Im} \frac{\mathbf{x} d\bar{\mathbf{x}}}{2r_0(x_0 + r_0)}, \\ \mathbf{A}_- &= \text{Im} \frac{\bar{\mathbf{w}} d\mathbf{w}}{\bar{\mathbf{w}}\mathbf{w} - 1} = -\text{Im} \frac{\bar{\mathbf{x}} d\mathbf{x}}{2r_0(x_0 + r_0)}. \end{aligned} \quad (6)$$

Let us prove it following the arguments of Atiah [9]. In quaternionic notation the strength of the  $SU(2)$  gauge field with potential  $\mathbf{A} = A_a \mathbf{e}_a/2$  is defined by the expression  $\mathbf{F} = d\mathbf{A} + \mathbf{A} \wedge \mathbf{A}$ . Hence, calculating it for the  $\mathbf{A}_\pm$ , we get

$$\begin{aligned} \mathbf{F}_+ &= -\frac{d\mathbf{w} \wedge d\bar{\mathbf{w}}}{(\mathbf{w}\bar{\mathbf{w}} - 1)^2} = \frac{\mathbf{e}_a \tilde{\omega}_a^{(2)+}}{2(1 - \mathbf{w}\bar{\mathbf{w}})^2}, \\ \mathbf{F}_- &= -\frac{d\bar{\mathbf{w}} \wedge d\mathbf{w}}{(\bar{\mathbf{w}}\mathbf{w} - 1)^2} = \frac{\mathbf{e}_a \tilde{\omega}_a^{(2)-}}{2(1 - \mathbf{w}\bar{\mathbf{w}})^2}, \end{aligned} \quad (7)$$

where

$$\begin{aligned} \tilde{\omega}_a^{(2)\pm} &= (\pm dw_4 \wedge dw_1 + dw_2 \wedge dw_3, \\ &\quad \pm dw_4 \wedge dw_2 + dw_3 \wedge dw_1, \\ &\quad \pm dw_4 \wedge dw_3 + dw_1 \wedge dw_2). \end{aligned} \quad (8)$$

The set  $\tilde{\omega}_a^+$  defines, precisely, the basis of self-dual two-forms, and  $\tilde{\omega}_a^-$  that of anti-self-dual ones. Consequently, the  $\mathbf{F}_+$  is a self-dual field, and  $\mathbf{F}_-$  is an anti-self-dual one:  $\star \mathbf{F}_\pm = \pm \mathbf{F}_\pm$ .

Seemingly,  $|\mathbf{F}^\pm| \rightarrow \infty$ , when  $|w| \rightarrow \infty$  (and  $|\mathbf{x}| \rightarrow \infty$ ). However, considering the classical motion of a particle the on four-dimensional hyperboloid in the presence of these fields (see [12]), one can see that the constructed instanton and anti-instanton configurations have a constant uniform magnitude. Indeed, the magnitude of the gauge field is defined as the strength multiplied on the inverse metrics. Hence, the product of the matrices  $\mathbf{F}_\pm$  given by (7) on the inverse to the metrics (5) is equal to the constant two-form  $-\mathbf{e}_a \tilde{\omega}_a^\pm / 8r_0^2$ .

Finally, let us conclude this section writing down the components of connections (6) in real coordinates

$$\begin{aligned} A_1^\pm &= 2 \frac{\pm w_4 dw_1 + w_3 dw_2 - w_2 dw_3 \mp w_1 dw_4}{w_\mu w_\mu - 1} \\ &= -\frac{\pm x_4 dx_1 + x_3 dx_2 - x_2 dx_3 \mp x_1 dx_4}{r_0(x_0 + r_0)}, \end{aligned} \quad (9)$$

$$\begin{aligned} A_2^\pm &= 2 \frac{-w_3 dw_1 \mp w_4 dw_2 - w_1 dw_3 \pm w_2 dw_4}{w_\mu w_\mu - 1} \\ &= -\frac{-x_3 dx_1 \mp x_4 dx_2 - x_1 dx_3 \pm x_2 dx_4}{r_0(x_0 + r_0)}, \end{aligned} \quad (10)$$

$$\begin{aligned} A_3^\pm &= 2 \frac{w_2 dw_1 - w_1 dw_2 \mp w_4 dw_3 \pm w_3 dw_4}{w_\mu w_\mu - 1} \\ &= -\frac{x_2 dx_1 - x_1 dx_2 \mp x_4 dx_3 \pm x_3 dx_4}{r_0(x_0 + r_0)}. \end{aligned} \quad (11)$$

### 3. Oscillator

The quantum mechanics of the colored particle moving on the four-dimensional hyperboloid in the presence of the potential  $V$  and (anti-)instanton field is described by the Hamiltonian

$$\hat{H} = -\frac{\hbar^2(1-w_\mu w_\mu)^4}{2r_0^2} \mathcal{D}_\mu (1-w_\mu w_\mu)^{-2} \mathcal{D}_\mu + V(w, \bar{w}),$$

$$\mathcal{D}_\mu = \partial/\partial w_\mu + i A_{(a)\mu} T_a, \quad (12)$$

where  $T_a$  are the  $SU(2)$  generators on the internal space  $S^2$  of the (anti-)instanton,  $[\hat{T}_a, \hat{T}_b] = i\epsilon_{abc} \hat{T}_c$ , and  $A_A^a$  is defined by (6).

We choose the oscillator potential on the four-dimensional hyperboloid given by the expression

$$V_{osc} = 2\omega^2 r_0^2 \mathbf{w} \bar{\mathbf{w}} = 2\omega^2 r_0^2 \frac{x_0 - r_0}{x_0 + r_0}, \quad (13)$$

which is similar to the one on the complex projective space  $\mathbb{CP}^1$  [2], and on the quaternionic projective spaces  $\mathbb{HP}^1$  [3,4]. Let us show that this oscillator system is an exactly solvable one and calculate its wavefunction and spectrum.

We restrict ourselves to the upper sheet of hyperboloid and introduce the “hyperspherical” coordinates

$$x_0 = r_0 \cosh \theta, \quad x_2 + ix_1 = r_0 \sinh \theta \sin \frac{\beta}{2} e^{i\frac{\alpha-\gamma}{2}},$$

$$x_4 + ix_3 = r_0 \sinh \theta \cos \frac{\beta}{2} e^{i\frac{\alpha+\gamma}{2}}, \quad (14)$$

or, equivalently,

$$w_2 + iw_1 = \tanh \frac{\theta}{2} \sin \frac{\beta}{2} e^{i\frac{\alpha-\gamma}{2}},$$

$$w_4 + w_3 = \tanh \frac{\theta}{2} \cos \frac{\beta}{2} e^{i\frac{\alpha+\gamma}{2}}, \quad (15)$$

where  $\theta \in [0, \infty)$ ,  $\beta \in [0, \pi]$ ,  $\alpha \in [0, 2\pi)$ ,  $\gamma \in [0, 4\pi)$ .

In these terms the quantum Hamiltonian with the oscillator potential (13) and with the instanton field  $A_a^+$  reads

$$\mathcal{H}^+ = -\frac{1}{2r_0^2} \left[ \frac{1}{\sinh^3 \theta} \frac{\partial}{\partial \theta} \left( \sinh^3 \theta \frac{\partial}{\partial \theta} \right) + \frac{2\hat{L}^2}{1 - \cosh \theta} \right. \\ \left. + \frac{2\hat{J}^2}{1 + \cosh \theta} \right] + 2\omega^2 r_0^2 \frac{\cosh \theta - 1}{\cosh \theta + 1}. \quad (16)$$

Here  $\hat{L}_a$  are the components of the  $SU(2)$  momentum  $[\hat{L}_a, \hat{L}_b] = i\epsilon_{abc} \hat{L}_c$ ,

$$\hat{L}_1 = i \left( \cos \alpha \cot \beta \frac{\partial}{\partial \alpha} + \sin \alpha \frac{\partial}{\partial \beta} - \frac{\cos \alpha}{\sin \beta} \frac{\partial}{\partial \gamma} \right),$$

$$\hat{L}_2 = i \left( \sin \alpha \cot \beta \frac{\partial}{\partial \alpha} - \cos \alpha \frac{\partial}{\partial \beta} - \frac{\sin \alpha}{\sin \beta} \frac{\partial}{\partial \gamma} \right),$$

$$\hat{L}_3 = -i \frac{\partial}{\partial \alpha} \quad (17)$$

and  $\hat{J}_a = \hat{L}_a + \hat{T}_a$ ,

$$[\hat{L}_a, \hat{L}_b] = i\epsilon_{abc} \hat{L}_c, \quad [\hat{L}_a, \hat{J}_b] = i\epsilon_{abc} \hat{L}_c,$$

$$[\hat{J}_a, \hat{J}_b] = i\epsilon_{abc} \hat{J}_c. \quad (18)$$

It is convenient to represent the generators  $T^a$  in terms of  $S^3$ , as in (17) (where, instead of  $\alpha, \beta, \gamma$ , there appear the coordinates of  $S^3$ ,  $\alpha_T, \beta_T, \gamma_T$ ), with the following condition imposed

$$\hat{T}^2 \Psi(\alpha, \beta, \gamma, \theta, \alpha_T, \beta_T, \gamma_T) \\ = T(T+1) \Psi(\alpha, \beta, \gamma, \theta, \alpha_T, \beta_T, \gamma_T), \quad (19)$$

which corresponds to the fixation of the isospin  $T$ . Notice that the generators  $\hat{J}_a, \hat{L}^2, \hat{T}^2$  are constants of motion, while  $\hat{L}_a, \hat{T}_a$  do not commute with the Hamiltonian.

In order to solve the Schrödinger equation  $\mathcal{H}\Psi = \mathcal{E}\Psi$ , we introduce the separation ansatz

$$\Psi(\theta, \alpha, \beta, \gamma, \alpha_T, \beta_T, \gamma_T) = Z(\theta) \Phi(\alpha, \beta, \gamma, \alpha_T, \beta_T, \gamma_T), \quad (20)$$

where  $\Phi$  are the eigenfunctions of  $\hat{L}^2, \hat{T}^2$  and  $\hat{J}^2$  with the eigenvalues  $L(L+1), T(T+1)$  and  $J(J+1)$ . Thus,  $\Phi$  can be represented in the form

$$\Phi = \sum_{M=m+t} (JM|L, m'; T, t') D_{mm'}^L(\alpha, \beta, \gamma) D_{tt'}^T(\alpha_T, \beta_T, \gamma_T), \quad (21)$$

where  $(JM|L, m'; T, t')$  are the Clebsh–Gordan coefficients and  $D_{mm'}^L$  and  $D_{tt'}^T$  are the Wigner functions.

Using the above separation ansatz, we get the following “radial” Schrödinger equation:

$$\frac{1}{\sinh^3 \theta} \frac{d}{d\theta} \left( \sinh^3 \theta \frac{dZ}{d\theta} \right) + \frac{2J(J+1)}{1 + \cosh \theta} Z + \frac{2L(L+1)}{1 - \cosh \theta} Z \\ + 2r_0^2 \left( \mathcal{E} - 2\omega^2 r_0^2 \frac{\cosh \theta - 1}{\cosh \theta + 1} \right) Z = 0. \quad (22)$$

Now, making the substitution  $Z(\theta) = \sinh^{-3/2} \theta R(\theta)$ , we end up with the equation

$$\frac{d^2 R}{d\theta^2} + \left[ \mathcal{E} - \frac{L(L+1) + 3/16}{\sinh^2 \theta/2} + \frac{\tilde{J}(\tilde{J}+1) + 3/16}{\cosh^2 \theta/2} \right] R = 0, \quad (23)$$

where we introduced the notation

$$\mathcal{E} = 2r_0^2 \mathcal{E} - 4\omega^2 r_0^4 - \frac{9}{4},$$

$$\tilde{J}(\tilde{J}+1) \equiv J(J+1) + 4\omega^2 r_0^2. \quad (24)$$

The same equation appears also in the Schrödinger equation of the Higgs oscillator on a 4-dimensional hyperboloid [13].

The regular solution of this equation is the hypergeometric function

$$R_{nJL} = \left( \sinh \frac{\theta}{2} \right)^{2L+3/2} \left( \cosh \frac{\theta}{2} \right)^{2n-2\tilde{J}-1/2} \\ \times {}_2F_1(-n, n+2\tilde{J}+1, 2L+2, \tanh^2 \theta/2), \quad (25)$$

where  $n = \sqrt{-\mathcal{E}} + \tilde{J} - L - 1/2$  is a nonnegative integer number  $n = 0, 1, 2, \dots, [\tilde{J} - L - 1/2]$ . Taking into account the expression (24), we get the energy spectrum of the system

$$\mathcal{E} = \frac{(\tilde{J} - J)(n + L + 1)}{r_0^2} - \frac{(n + L - J - 1)(n + L - J + 2)}{2r_0^2}. \quad (26)$$

The regular normalized wavefunction is defined by the expression

$$Z(\theta) = \sqrt{\frac{(2\tilde{J} - 2L - 2n - 1)(n + 2L + 1)! \Gamma(2\tilde{J} - n + 1)}{n! \Gamma(2\tilde{J} - 2L - n)}} \times (\sinh \theta)^{-3/2} R_{nJL}. \quad (27)$$

Let us remind that

$$J = |L - T|, |L - T| + 1, \dots, L + T, \\ n = 0, 1, 2, \dots, [\tilde{J} - L - 1/2], \quad L = 0, 1/2, 1, \dots \quad (28)$$

In the absence of the instanton field one has  $\tilde{J} = J$ . In this case one can introduce the principal quantum number  $\mathcal{N} = n + J + L$ , and get standard expressions for the spectrum and wavefunctions of the oscillator on the four-dimensional hyperboloid.

**Remark 1.** A similar system with the anti-instanton field is described by the Hamiltonian

$$\mathcal{H}^- = \frac{1}{2r_0^2} \left[ \frac{1}{\sinh^3 \theta} \frac{\partial}{\partial \theta} \left( \sinh^3 \theta \frac{\partial}{\partial \theta} \right) + \frac{2\hat{L}^2}{1 + \cosh \theta} + \frac{2\hat{J}^2}{1 - \cosh \theta} \right] + 2\omega^2 r_0^2 \frac{\cosh \theta - 1}{\cosh \theta + 1}. \quad (29)$$

Hence, its spectrum and wavefunctions can be obtained from the above ones, (26), (27) by the redefinition

$$\tilde{L} \rightarrow \tilde{J}, \quad J \rightarrow L, \\ \tilde{J}(\tilde{J} + 1) \equiv J(J + 1) + 2\omega^2 r_0^2. \quad (30)$$

**Remark 2.** The above results could be easily extended to the system with the “singular oscillator” potential defined as follows:

$$V^{so} = 2\omega^2 r_0^2 \frac{\cosh \theta - 1}{\cosh \theta + 1} + 2\omega_1^2 r_0^2 \frac{\cosh \theta + 1}{\cosh \theta - 1}. \quad (31)$$

The spectrum and wavefunctions of this system with the instanton field can be obtained from (26), (27) by the redefinition

$$J \rightarrow \tilde{J}, \quad \tilde{J}(\tilde{J} + 1) \equiv J(J + 1) + 2\omega_1^2 r_0^2, \\ \mathcal{E} \rightarrow \mathcal{E} - 2\omega_1^2 r_0^2. \quad (32)$$

Similarly, for the anti-instanton configuration we should make the following substitution:

$$L \rightarrow \tilde{L}, \quad \tilde{L}(\tilde{L} + 1) \equiv L(L + 1) + 2\omega_1^2 r_0^2, \\ \mathcal{E} \rightarrow \mathcal{E} - 2\omega_1^2 r_0^2. \quad (33)$$

Hence, the instantonic singular oscillator with “characteristic frequencies”  $(\omega, \omega_1)$  is “isomorphic” to the anti-instantonic singular oscillator with “characteristic frequencies”  $(\omega_1, \omega)$ .

#### 4. Discussion

We constructed the hyperbolic analogs of the BPST (anti-)instanton and of the oscillator potential, which preserve the exact solvability of the particle moving on a four-dimensional

hyperboloid in their presence. We calculated the energy spectrum and the wavefunctions of this model and found, that it possesses a degenerate ground state. Hence, we suggest that this system could form the appropriate ground for developing the relativistic theory of the higher-dimensional Hall effect. The system inherits the asymmetry with respect to instanton and anti-instanton fields, earlier observed in the models on the four-dimensional plane [8] and sphere [4]. Also, it has a finite discrete energy spectrum, which is typical for the systems on spaces with constant negative curvature. Notice that the suggested system could be viewed as a spherical part of the quantum-mechanical system on the  $\mathbb{R}^{4,1}$  describing the motion of a particle interacting with the “hyperbolic Yang monopole” (6) and potential (13), where  $r_0$  is a dynamical variable. Hence, it could be obtained, by the reduction associated with the second Hopf map, from the appropriate systems on  $\mathbb{R}^{4,4}$  and on the  $\mathcal{L}_3 = SU(3,1)/U(2)$  (compare, respectively, with [15] and [14]), which are specified by the absence of external gauge fields. Finally, let us mention that there is a kind of duality between monopoles and relativistic spinning particles (at the moment it is part of a folklore in physics, but probably for the first time it was pointed out in [16]). From this viewpoint the nonrelativistic particle moving on the hyperboloid in the presence of an instanton field is dual to the  $(4 + 1)$ -dimensional massive spinning particle, similarly to the duality between a nonrelativistic particle moving on the two-dimensional hyperboloid in the presence of a constant magnetic field, and the free massive relativistic  $(2 + 1)$ -dimensional particle [17]. A possible direction for future developments is the consideration of supersymmetric extensions, as it was done in the first Ref. [2] and [18].

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